Supplemental material for 'Communication Rate Analysis for Event-based State Estimation'

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This documents provides additional material for our work on 'Communication Rate Analysis for Event-based State Estimation'. As this document is tightly interwoven with the original work, we will start with the numbering of equations where we left off. Consequently every equation with a numbering less or equal to 40 can be found in [1] while all other equations can be found herein.

This work has two main purposes. Firstly, we will provide a derivation of the normalization factor Z_0 in (22). Secondly, we will provide an closed form expression for the precision $\Lambda_{1:k}$ required in (30).

1 Derivation of the normalization factor Z_0

The normalization factor Z_0^k in (22) is given by

$$Z_0^k = \frac{1}{\sqrt{(2\pi)^k |\Sigma_{1:k}|}} \tag{41}$$

We are therefore concerned with obtaining an expression for $|\Sigma_{1:k}|$ (as defined by (14)) that only depends on the system parameters in (1). To do so, we will first derive the Cholesky decomposition $\Sigma_{1:k} = L_{1:k}^T L_{1:k}$. Using Assumption 2 we know that $P_k^+ = P_s, \forall k$. Using the Riccati equation

$$AP_\mathrm{s}A^\mathrm{T} - P_\mathrm{s} - AP_\mathrm{s}C^\mathrm{T}(CP_\mathrm{s}C^\mathrm{T} + R)^{-1}CP_\mathrm{s}A^\mathrm{T} + Q = 0. \tag{42}$$

we will proof by induction that $L_{1:k} \in \mathbb{R}^{mk\times mk}$ is a block matrix with k^2 blocks of size $m \times m$ given by

$$L_{1:k} = \begin{pmatrix} \alpha & \Delta_1 & \Delta_2 & \Delta_3 & \cdots & \Delta_{k-1} \\ & \alpha & \Delta_1 & \Delta_2 & \cdots & \Delta_{k-2} \\ & & \alpha & \Delta_1 & \cdots & \Delta_{k-3} \\ & & & \ddots & & \vdots \\ & \text{\Large 0} & & & \alpha & \Delta_1 \\ & & & & & \alpha \end{pmatrix} \qquad \begin{aligned} &\text{with} \\ &\alpha = \left(CP_sC^{\mathrm{T}} + R\right)^{\frac{1}{2}} \\ &\Delta_i = \alpha^{-\mathrm{T}} CP_s \left(A^{\mathrm{T}}\right)^i C^{\mathrm{T}} \end{aligned} \tag{43}$$

Base Case Starting with $n = 1$ it's trivial to see that

$$\Sigma_{1:1} = \left[CP_sC + R\right] = \left[\alpha^{\mathrm{T}}\alpha\right] = L_{1:1}^T L_{1:1} \tag{44}$$

such that the statement is true. Repeating the calculations for $n = 2$ yields

$$\begin{aligned} \Sigma_{1:2} &= \begin{bmatrix} CP_sC^{\mathrm{T}} + R & CP_sA^{\mathrm{T}}C^{\mathrm{T}} \\ CAP_sC^{\mathrm{T}} & CAP_sA^{\mathrm{T}}C^{\mathrm{T}} + CQC^{\mathrm{T}} + R \end{bmatrix} \\ L_{1:2}^T L_{1:2} &= \begin{bmatrix} \alpha^{\mathrm{T}} & 0 \\ \Delta_1^{\mathrm{T}} & \alpha^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \alpha & \Delta_1 \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} \alpha^{\mathrm{T}}\alpha & \alpha^{\mathrm{T}}\Delta_1 \\ \Delta_1^{\mathrm{T}}\alpha & \alpha^{\mathrm{T}}\alpha + \Delta_1^{\mathrm{T}}\Delta_1 \end{bmatrix} \\ &= \begin{bmatrix} CP_sC^{\mathrm{T}} + R & CP_sA^{\mathrm{T}}C^{\mathrm{T}} \\ CAP_sC^{\mathrm{T}} & CP_sC^{\mathrm{T}} + R + CAP_sC^{\mathrm{T}}\left(CP_sC^{\mathrm{T}} + R\right)^{-1} CP_sA^{\mathrm{T}}C^{\mathrm{T}} \end{bmatrix} \end{aligned} \tag{45}$$

equality of all elements but the one in the lower right corner follow trivially. For the remaining element we can apply the Riccati equation (42) to yield

$$\begin{aligned} &CP_sC^{\mathrm{T}} + R + CAP_sC^{\mathrm{T}}\left(CP_sC^{\mathrm{T}} + R\right)^{-1} CP_sA^{\mathrm{T}}C^{\mathrm{T}} \\ &\quad = C\left(P_s + AP_sC^{\mathrm{T}}\left(CP_sC^{\mathrm{T}} + R\right)^{-1} CP_sA^{\mathrm{T}}\right) C^{\mathrm{T}} + R \\ &\quad = CAP_sA^{\mathrm{T}}C^{\mathrm{T}} + CQC^{\mathrm{T}} + R \end{aligned} \tag{46}$$

which shows that the statement is true for the base cases with $n = 1, 2$.

Induction Step In the induction step the horizontal and vertical expansion of the matrix $\Sigma_{1:k}$ is analyzed. We establish a relationship between $\Sigma_{1:k}$ and $\Sigma_{1:k+1}$. Assuming statement (43) is true for $\Sigma_{1:k}$ we show the statement to be true for $\Sigma_{1:k+1}$ as well. We use the notation $\left(\Sigma_{1:k}\right)_{i,j}$ to denote the block matrix of size $m \times m$ in the i-th vertical position and j-th horizontal position of the matrix $\Sigma_{1:k}$.

We first inspect the relationship between $\Sigma_{1:k}$ and $\Sigma_{1:k+1}$. Using (12), (14) we can rewrite $(\Sigma_{1:k})_{i,j}$ as

$$(\Sigma_{1:k})_{i,j} = C\rho_{i,j}C^{\mathrm{T}} + R\delta_{i,j} \quad \text{with} \quad \rho_{i,j} \coloneqq A^i P_s \left(A^j\right)^{\mathrm{T}} + \sum_{k=1}^{i} A^{i-k} Q \left(A^{j-k}\right)^{\mathrm{T}} \tag{47}$$

For the relationship $\Sigma_{1:k} \to \Sigma_{1:k+1}$ we find that for $i \leq j$

$$\begin{aligned} (\Sigma_{1:k})_{i+1,j} &= CA\rho_{i,j}C^{\mathrm{T}} + Q\left(A^{j-i-1}\right)^{\mathrm{T}} + \delta_{i+1,j}R \\ (\Sigma_{1:k})_{i,j+1} &= C\rho_{i,j}A^{\mathrm{T}}C^{\mathrm{T}}, \quad \text{for } i < j. \end{aligned} \tag{48}$$

In the following proof, we will show that based on assumption that the form (43) holds for a given j, i it also holds true for any $j+1, i+1$.

Product $L^T L$: Since $L_{1:k}$ is an upper triangular matrix, the product $L_{1:k}^{\mathrm{T}} L_{1:k}$ with $i \leq j$ can be simplified simplified as

$$\begin{aligned} \left(L_{1:k}^{\mathrm{T}} L_{1:k}\right)_{i,j} &= \sum_{n=1}^{k} \left(L_{1:k}^{\mathrm{T}}\right)_{i,n} (L_{1:k})_{n,j} \\ &= \sum_{n=1}^{k} (L_{1:k})_{n,i}^{\mathrm{T}} (L_{1:k})_{n,j} \\ &= \sum_{n=1}^{i-1} (L_{1:k})_{n,i}^{\mathrm{T}} (L_{1:k})_{n,j} + (L_{1:k})_{i,i}^{\mathrm{T}} (L_{1:k})_{i,j} \end{aligned} \tag{49}$$

We continue to derive the expression for $\left(L_{1:k}^{\mathrm{T}} L_{1:k}\right)_{i+1,j}$ and $\left(L_{1:k}^{\mathrm{T}} L_{1:k}\right)_{i,j+1}$. We show that assuming the form (43) satisfies equation 48. We know that $(\Sigma_{1:k})_{i,j} = (\Sigma_{1:k+1})_{i,j}$ for $i, j \leq k$. From (49), one can see that it holds that $(L_{1:k})_{i,j} = (L_{1:k+1})_{i,j}$. Therefore it's sufficient to show (43) for $1 \leq i \leq k, j = k+1$ and $i = j = k+1$. The case $1 \leq j \leq k, i = k+1$ follows from the first case by symmetry of the covariance. To simplify notation we will set $L \coloneqq L_{1:k+1}$ in the following.

Part 1: Vertical Expansion with $i+1 < j$: We start by showing the $\left(L^{\mathrm{T}} L\right)_{i+1,j} = \Sigma_{i+1,j}$ for $i+1 < j$. We know that $\delta_{i,j} = \delta_{i+1,j} = 0$. From the induction assumption, we know the following relationship to be true

$$
\begin{aligned}
\left(L^{\mathrm{T}}L\right)_{i,j} &= \sum_{n=1}^{i-1} L_{n,i}^{\mathrm{T}} L_{n,j} + L_{i,i}^{\mathrm{T}} L_{i,j} \\
&= C\left\{\sum_{n=1}^{i-1} A^{i-k} P_s C^{\mathrm{T}} \left(C P_s C^{\mathrm{T}} + R\right)^{-1} C P_s \left(A^{\mathrm{T}}\right)^{j-k} + P_s \left(A^{\mathrm{T}}\right)^{j-i}\right\} C^{\mathrm{T}} \\
&= C \rho_{i,j} C^{\mathrm{T}} \qquad (50)
\end{aligned}
$$

Assuming structure (43) for the new elements $L_{i+1,i+1}$ we obtain for $\left(L^{\mathrm{T}}L\right)_{i+1,j}$

$$
\begin{aligned}
\left(L^{\mathrm{T}}L\right)_{i+1,j} &= \sum_{n=1}^{i} L_{n,i+1}^{\mathrm{T}} L_{n,j} + L_{i+1,i+1}^{\mathrm{T}} L_{i+1,j} \\
&= C\left\{\sum_{n=1}^{i} A^{i+1-k} P_s C^{\mathrm{T}} \left(C P_s C^{\mathrm{T}} + R\right)^{-1} C P_s \left(A^{\mathrm{T}}\right)^{j-k} + P_s \left(A^{\mathrm{T}}\right)^{j-i-1}\right\} C^{\mathrm{T}} \\
&= C\left\{A\rho_{i,j} - A P_s \left(A^{\mathrm{T}}\right)^{j-i} + A P_s C^{\mathrm{T}} \left(C P_s C^{\mathrm{T}} + R\right)^{-1} C P_s \left(A^{\mathrm{T}}\right)^{j-i}\right. \\
&\qquad \left. + P_s \left(A^{\mathrm{T}}\right)^{j-i-1}\right\} C^{\mathrm{T}} \qquad (51)
\end{aligned}
$$

We can apply the Riccati equation to (51) and obtain

$$
-\left\{A P_s A^{\mathrm{T}} + A P_s C^{\mathrm{T}} \left(C P_s C^{\mathrm{T}} + R\right)^{-1} C P_s A^{\mathrm{T}} + P_s\right\} \left(A^{\mathrm{T}}\right)^{j-i-1} = Q \left(A^{\mathrm{T}}\right)^{j-i-1} \qquad (52)
$$

Combining equation (52) with (51) yields the required result (48).

Part 2: Vertical Expansion with $i+1=j$: We now complement the proof of vertical expansion by showing that $\left(L^{\mathrm{T}}L\right)_{i+1,j} = \Sigma_{i+1,j}$ for $i+1=j$. Assuming structure (43) for the new elements $(L_{1:k+1})_{i+1,i+1}$ we obtain for $\left(L^{\mathrm{T}}L\right)_{i+1,j}$

$$
\begin{aligned}
\left(L^{\mathrm{T}}L\right)_{i+1,i+1} &= \sum_{n=1}^{i} L_{n,i+1}^{\mathrm{T}} L_{n,j} + L_{i+1,i+1}^{\mathrm{T}} L_{i+1,j} \\
&= C\left\{\sum_{n=1}^{i} A^{i+1-k} P_s C^{\mathrm{T}} \left(C P_s C^{\mathrm{T}} + R\right)^{-1} C P_s \left(A^{\mathrm{T}}\right)^{j-k} + P_s\right\} C^{\mathrm{T}} + R \\
&= C\left\{A\rho_{i,j} - A P_s \left(A^{\mathrm{T}}\right) + A P_s C^{\mathrm{T}} \left(C P_s C^{\mathrm{T}} + R\right)^{-1} C P_s \left(A^{\mathrm{T}}\right) + P_s\right\} C^{\mathrm{T}} + R \\
&= C\left\{A\rho_{i,j} - Q\right\} C^{\mathrm{T}} + R \qquad (53)
\end{aligned}
$$

where we applied again the Riccati equation (42) to show the required result (48).

Part 3: Horizontal Expansion We show that $\left(L^{\mathrm{T}}L\right)_{i,j+1} = \Sigma_{i,j+1}$ for $i < j$. The case $j = i$ is irrelevant for the horizontal expansion, since it was shown previously in the vertical expansion with $i = j$. Assuming the same structure for the new elements $L_{i+1,i+1}$ we obtain for $\left(L^{\mathrm{T}}L\right)_{i+1,j}$

$$\begin{aligned}\left(L^{\mathrm{T}}L\right)_{i,j+1} &= \sum_{n=1}^{i-1} L_{n,i}^{\mathrm{T}} L_{n,j+1} + L_{i,i}^{\mathrm{T}} L_{i,j+1} \\ &= C\left\{\sum_{n=1}^{i-1} A^{i-k} P_s C^{\mathrm{T}} \left(C P_s C^{\mathrm{T}} + R\right)^{-1} C P_s \left(A^{\mathrm{T}}\right)^{j+1-k} + P_s \left(A^{\mathrm{T}}\right)^{j+1-i}\right\} C^{\mathrm{T}} \\ &= C\left\{\rho_{i,j} A^{\mathrm{T}}\right\} C^{\mathrm{T}} \qquad (54)\end{aligned}$$

where (54) also has the required form (48).

Summary We thus showed the existence of a solution of form (43) for $\Sigma_{1:k+1}$. The uniqueness of the solution follows from the uniqueness of the Cholesky decomposition for positive definite matrices, which concludes the proof.

Finally, we can use form (43) to obtain an expression for $|\Sigma_{1:k}|$. As a determinant of a block triangular matrix is given by the product of its diagonal elements [2], we can see that

$$|\Sigma_{1:k}| = \alpha^k \quad \text{with} \quad \alpha = (C P_s C^{\mathrm{T}} + R)^{\frac{1}{2}} \qquad (55)$$

Consequently it follows for the normalization factor (41) $Z_0 \coloneqq (2\pi\alpha)^{-\frac{1}{2}}$.

1.1 Precision matrix $\Lambda_{1:k}$

In this section we derive an expression for the precision $\Lambda_{1:k}$. We will use the result (43) to rewrite the precision using its Cholesky decomposition $\Lambda_{1:k} = \Sigma_{1:k}^{-1} = (L_{1:k}^{\mathrm{T}} L_{1:k})^{-1} = L_{1:k}^{-1} L_{1:k}^{-\mathrm{T}}$. It's sufficient to show that $L_{1:k}^{-1}$ has a closed form expression. We now show by proof of induction that L^{-1} can be written as

$$L_{1:k}^{-1} = \begin{pmatrix} \beta & \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_{k-1} \\ & \beta & \zeta_0 & \zeta_1 & \cdots & \zeta_{k-2} \\ & & \beta & \zeta_1 & \cdots & \zeta_{k-3} \\ & & & \ddots & & \vdots \\ & \mathbf{0} & & & \beta & \zeta_1 \\ & & & & & \beta \end{pmatrix} \qquad \begin{aligned} &\text{with} \\ &\beta = \alpha^{-1} = \left(CP_sC^{\mathrm{T}} + R\right)^{-\frac{1}{2}} \\ &\Xi := -\left(CP_sC^{\mathrm{T}} + R\right)^{-1} CP_sA^{\mathrm{T}} \\ &\xi := A + C^{\mathrm{T}}\Xi \\ &\zeta_i = \Xi\xi^i C^{\mathrm{T}}\alpha^{-1} \end{aligned} \tag{56}$$

In the following we will usually drop the index $1:k$ and simply write L instead of $L_{1:k}$. Whenever it's necessary to emphasize the precise index, we will fall back to the elaborate form. Since L and L^{-1} are both upper triangular matrices, the product of L^{-1} and L can be expressed as

$$\left(L^{-1}L\right)_{i,j} = \sum_{k=1}^{n} L_{i,k}^{-1}L_{k,j} = \sum_{k=i}^{j} L_{i,k}^{-1}L_{k,j}. \tag{57}$$

From (57), one can easily compute the diagonal elements of L^{-1}. We find that

$$\left(L^{-1}L\right)_{i,i} = L_{i,i}^{-1}L_{i,i} \overset{!}{=} \mathrm{I}_k \Rightarrow L_{i,i}^{-1} = (L_{i,i})^{-1} = \alpha^{-1} \tag{58}$$

where $\mathrm{I}_k \in \mathbb{R}^{k\times k}$ denotes the identity matrix. Next, We show the diagonal structure of the matrix for non-diagonal elements, i.e. $L_{i+1,j+1}^{-1} = L_{i,j}^{-1}$ with $j > i$

$$\begin{aligned} (L^{-1}L)_{i+1,j+1} &= \sum_{k=i+1}^{j+1} L_{i+1,k}^{-1}L_{k,j+1} = \sum_{k=i}^{j} L_{i+1,k+1}^{-1}L_{k+1,j+1} \\ &= \sum_{k=i}^{j} L_{i+1,k+1}^{-1}L_{k,j} \overset{!}{=} 0 \end{aligned} \tag{59}$$

where we used $L_{i+1,j+1} = L_{i,j}$ from (43). With $(LL^{-1})_{i,j} = 0$ it follows

$$(L^{-1}L)_{i+1,j+1} - (L^{-1}L)_{i,j} = \sum_{k=i}^{j} \left(L_{k+1,j+1}^{-1} - L_{k,j}^{-1}\right) L_{i,k} \overset{!}{=} 0 \tag{60}$$

Since (60) holds for all i, j with $j > i$, we follow that indeed $L_{k,j}^{-1} = L_{k+1,j+1}^{-1}$. Finally, we show (56) for $j > i$ by proof of induction.

Induction start From (58) we know $L_{1:1}$ to satisfy form (56).

Induction step For the induction step we assume (56) to hold for $(L_{1:k})_{i,j}$ with $j > i$. From (57), we see that $\left(L_{1:k}^{-1}\right)_{i,j} = \left(L_{1:k+1}^{-1}\right)_{i,j}$ for $1 \leq i \leq k, 1 \leq j \leq k$. Next, we show the statement to hold true for $(L_{1:k+1})_{j+1}$. We solve $\left(L^{-1}L\right)_{i,j+1} \stackrel{!}{=} 0$ for $L_{i,j+1}^{-1}$ and show it to satisfy form (56). It holds true that

$$\begin{aligned}
\left(L^{-1}L\right)_{i,i+1} &= L_{i,i}^{-1}L_{i,i+1} + L_{i,i+1}L_{i+1,i+1}^{-1} = \alpha^{-1}L_{i,i+1} + L_{i,i+1}^{-1}\alpha \stackrel{!}{=} 0 \\
L_{i,i+1}^{-1} &= -\alpha^{-1}L_{i,i+1}\alpha^{-1} = -\left(CP_sC^{\mathrm{T}} + R\right)^{-1} CP_sA^{\mathrm{T}}C^{\mathrm{T}}\alpha^{-1} \\
&= \Xi\xi^0 C^{\mathrm{T}}\alpha^{-1}
\end{aligned} \tag{61}$$

Per assumption we know $L_{i,j}^{-1}$ to satisfy the following form

$$\begin{aligned}
L_{i,j}^{-1} &= -\left(\sum_{k=i+1}^{j-1} L_{i,k}^{-1}L_{k,j} + \alpha^{-1}L_{i,j}\right)\alpha^{-1} \\
&= \left(\left[\sum_{i+1}^{j-1} \Xi\xi^{k-i-1}C^{\mathrm{T}}\Xi\left(A^{\mathrm{T}}\right)^{j-k-1}C^{\mathrm{T}}\right] + \Xi\left(A^{\mathrm{T}}\right)^{j-i-1}C^{\mathrm{T}}\right)\alpha^{-1} \\
&= \Xi\xi^{j-i-1}C^{\mathrm{T}}\alpha^{-1}
\end{aligned} \tag{62}$$

We find for $L_{i,j+1}^{-1}$

$$\begin{aligned}
L_{i,j+1}^{-1} &= -\left(\sum_{k=i+1}^{j} L_{i,k}^{-1}L_{k,j+1} + \alpha^{-1}L_{i,j+1}\right)\alpha^{-1} \\
&= \left(\left[\sum_{i+1}^{j} \Xi\xi^{k-i-1}C^{\mathrm{T}}\Xi\left(A^{\mathrm{T}}\right)^{j-k}C^{\mathrm{T}}\right] + \Xi\left(A^{\mathrm{T}}\right)^{j-i}C^{\mathrm{T}}\right)\alpha^{-1} \\
&= \left(\left[\sum_{i+1}^{j-1} \Xi\xi^{k-i-1}C^{\mathrm{T}}\Xi\left(A^{\mathrm{T}}\right)^{j-k-1}C^{\mathrm{T}}\right]A^{\mathrm{T}} + \Xi\xi^{k-i-1}C^{\mathrm{T}}\Xi C^{\mathrm{T}} + \Xi\left(A^{\mathrm{T}}\right)^{j-i}C^{\mathrm{T}}\right)\alpha^{-1} \\
&= \Xi\xi^{j-i-1}\left(A^{\mathrm{T}} + C^{\mathrm{T}}\Xi\right)C^{\mathrm{T}}\alpha^{-1} \\
&= \Xi\xi^{j-i}C^{\mathrm{T}}\alpha^{-1}
\end{aligned} \tag{63}$$

which finishes the induction step.

References

[1] S. Ebner and S. Trimpe, “Communication rate analysis for event-based state estimation,” 2016.

[2] D. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas (Second Edition)*, ser. Princeton reference. Princeton University Press, 2009.