A Limiting Property of the Matrix Exponential

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***Abstract*—A limiting property of the matrix exponential is proven: if the (1,1)-block of a 2-by-2 block matrix becomes "arbitrarily small" in a limiting process, the matrix exponential of that matrix tends to zero in the (1,1)-, (1,2)-, and (2,1)-blocks. The limiting process is such that either the log norm of the (1,1)-block goes to negative infinity, or, for a certain polynomial dependency, the matrix associated with the largest power of the variable that tends to infinity is stable. The limiting property is useful for simplification of dynamic systems that exhibit modes with sufficiently different time scales. The obtained limit then implies the decoupling of the corresponding dynamics.**

***Index Terms*—Matrix exponential, limiting property, logarithmic norm, time-scale separation.**

I. INTRODUCTION

The subject of study in this paper is the matrix exponential

$$\exp\left(\begin{bmatrix} A_{11}-K(\alpha) & A_{12} \\ A_{21} & A_{22}\end{bmatrix} t\right), \quad t>0, \tag{1}$$

in the limit as $K(\alpha)$ grows large for $\alpha \to \infty$ in some sense to be made precise later. All matrices are complex, and α is a real parameter. For different classes of $K(\alpha)$, we derive sufficient (and in one case also necessary) conditions on $K(\alpha)$ such that, for all $t>0$,

$$\lim_{\alpha\to\infty} \exp\left(\begin{bmatrix} A_{11}-K(\alpha) & A_{12} \\ A_{21} & A_{22}\end{bmatrix} t\right) = \begin{bmatrix} 0 & 0 \\ 0 & e^{A_{22}t}\end{bmatrix}. \tag{2}$$

That is, we are interested in conditions guaranteeing that the coupling blocks (1,2) and (2,1) vanish (in addition to the (1,1)-block).

In addition to being an interesting matrix problem, the result can be applied to control systems that exhibit significantly different time scales, such as systems with high-gain feedback on some states. For example, consider the system

$$\dot{x}_{\mathrm{f}}(t) = A_{11}x_{\mathrm{f}}(t) + A_{12}x_{\mathrm{s}}(t) + u(t) \tag{3}$$
$$\dot{x}_{\mathrm{s}}(t) = A_{21}x_{\mathrm{f}}(t) + A_{22}x_{\mathrm{s}}(t), \tag{4}$$

with static feedback on the states $x_{\mathrm{f}}(t)$ (index f for "fast" and s for "slow"),

$$u(t) = -K(\alpha)x_{\mathrm{f}}(t). \tag{5}$$

The matrix function $K(\alpha)$ then represents the feedback gain parametrized by α. The feedback system is depicted in Fig. 1. A more general multi-loop feedback system with additional reference inputs is considered in [1].

The matrix exponential (1) is a fundamental matrix (see e.g. [2]) of the feedback system (3)–(5). The limit (2) means that the dynamics of $x_{\mathrm{f}}(t)$ and $x_{\mathrm{s}}(t)$ are decoupled in the limit as $K(\alpha)$ grows large. In this context, we seek to determine what type of feedback yields a decoupling of the states in feedback from the remaining ones in the limit as the feedback gains become arbitrarily large.

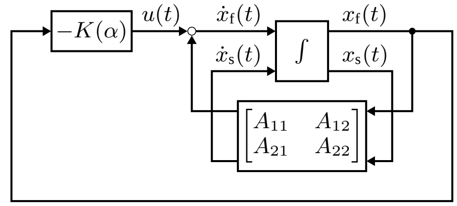

Fig. 1. Linear system with feedback on the first part of the state vector, the "fast" states x_{f}.

This question is of interest, for example, when designing multi-loop control systems with high-gain inner loops, since a decoupling of the states allows for a simplified system description and, hence, a simplified control design. The matrix result herein is applied in [1] to derive a time-scale separation algorithm for a cascaded control system with high-gain inner feedback loops. The algorithm yields a system description that includes the plant dynamics and the effect of the inner feedback loops. The obtained representation is useful, for example, for designing an outer-loop controller. This methodology is applied in the design of a cascaded feedback control system for an inverted pendulum in [1] and for a balancing cube (a multi-body 3-D inverted pendulum) in [3].

Related to the problem studied herein is the work by Campbell et al., [4], [5]. The authors consider the matrix exponential with its argument being a polynomial in $1/\varepsilon$ and derive conditions for its convergence in the limit as $\varepsilon \to 0^+$. In [4], for example, Campbell et al. present a necessary and sufficient condition for pointwise convergence of

$$\exp((A + B/\varepsilon)t), \quad t > 0, \tag{6}$$

as $\varepsilon \to 0^+$. While they are interested in general convergence to *some* limit, we seek conditions that yield the *particular* limit (2); that is, where the cross coupling blocks (1,2) and (2,1) vanish.

Before deriving the technical results, this article continues with notation and preliminaries in Sec. II. In Sec. III, we establish lemmas and cite theorems that are required for the development of the main results, which follows in Sec. IV. The main results are three different conditions on $K(\alpha)$ that guarantee (2): a sufficient condition that is based on the log norm (defined in (8) of the next section) of $-K(\alpha)$ and that makes no prior assumption on the function type of $K(\alpha)$ (Theorem 3); a necessary and sufficient condition for the case when $K(\alpha)$ is affine (Theorem 4); and another sufficient condition for the case when $K(\alpha)$ has an affine term and an additional polynomial term α^r, $r \geq 2$ (Theorem 5). The latter two results are based on [4], [5], while the first one is established independently of those. Numerical examples illustrating the applicability of the different theorems are given throughout in Sec. IV. The article concludes with remarks in Sec. V.

A preliminary version of Theorem 3 was first published in [1].

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II. Notation and Preliminaries

We use $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}^+$ to denote real numbers, complex numbers, and nonnegative real numbers, respectively. For the derivations in the paper, we work exclusively with the vector 2-norm and its induced matrix norm; that is, for $x \in \mathbb{C}^n$ and $A \in \mathbb{C}^{p\times n}$,

$$\|x\| = \Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2}, \quad \|A\| = \max_{\|x\|=1} \|Ax\|. \tag{7}$$

For $A \in \mathbb{C}^{n\times n}$, $\mu(A)$ denotes the *log norm* of A, [6]–[8]:

$$\mu(A) := \max\{\mu | \mu \text{ an eigenvalue of } (A + A^*)/2\}, \tag{8}$$

where A^* is the conjugate transpose of A. We shall exploit the following properties of $\mu(A)$, [6]–[8]:

$$\|e^{At}\| \le e^{\mu(A)t} \tag{9}$$

$$\mu(A) \le \|A\| \tag{10}$$

$$\mu(A + B) \le \mu(A) + \|B\|, \tag{11}$$

where $A, B \in \mathbb{C}^{n\times n}$ and $t \in \mathbb{R}^+$.

Let $\operatorname{spec}(A)$ denote the *spectrum of* $A \in \mathbb{C}^{n\times n}$ (the set of all eigenvalues of A ignoring algebraic multiplicity), and let OLHP denote the open left half plane in $\mathbb{C}$ (i.e. $\text{OLHP} := \{x \in \mathbb{C} : \operatorname{Re} x < 0\}$), [7]. The matrix A is called *stable* if $\operatorname{spec}(A) \subset \text{OLHP}$; and it is called *semistable* if $\operatorname{spec}(A) \subset \text{OLHP} \cup \{0\}$ and, if $0 \in \operatorname{spec}(A)$, then 0 is semisimple (i.e. its algebraic and geometric multiplicity are identical), [7]. The *index* of A, denoted $\operatorname{Index} A$, is the smallest nonnegative integer j such that $\operatorname{rank} A^j = \operatorname{rank} A^{j+1}$, [7]. The *Darzin inverse* of A is the unique matrix A^{D} satisfying $AA^{\mathrm{D}} = A^{\mathrm{D}}A$, $A^{\mathrm{D}}AA^{\mathrm{D}} = A^{\mathrm{D}}$, and $A^{j+1}A^{\mathrm{D}} = A^j$ with $j = \operatorname{Index} A$, [7], [9]. For $A, B \in \mathbb{C}^{n\times n}$, define $[A; B] := (I - B^{\mathrm{D}}B)A(I - B^{\mathrm{D}}B)$, [5], where I is the identity matrix.

The following two facts are useful in later derivations; the proofs are straightforward and therefore omitted.

Fact 1: Consider the matrix differential equation

$$\dot Z(t) = A\,Z(t) + B\,U(t),\ t \ge 0,\ Z(0) = Z_0, \tag{12}$$

where $Z : \mathbb{R}^+ \to \mathbb{C}^{n\times p}$ continuously differentiable, $U : \mathbb{R}^+ \to \mathbb{C}^{m\times p}$ continuous, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, and $Z_0 \in \mathbb{C}^{n\times p}$. The unique solution of (12) is

$$Z(t) = e^{At}Z_0 + \int_0^t e^{A(t-\tau)}\,B\,U(\tau)\,d\tau. \tag{13}$$

Fact 2: Let $A : [a, b] \to \mathbb{C}^{n\times m}$ be continuous. Then

$$\Big\|\int_a^b A(t)\,dt\Big\| \le \int_a^b \|A(t)\|\,dt. \tag{14}$$

III. Lemmas

This section establishes preliminary lemmas and restates two theorems from [4], [5], which are used in Sec. IV to prove the main results of this paper.

Consider the matrix ordinary differential equation (ODE):

$$\dot X(t) = (A_{11} - K(\alpha))X(t) + A_{12}Y(t), \quad X(0) = X_0 \tag{15}$$

$$\dot Y(t) = A_{21}X(t) + A_{22}Y(t), \quad Y(0) = Y_0 \tag{16}$$

with $X : \mathbb{R}^+ \to \mathbb{C}^{n\times p}$ and $Y : \mathbb{R}^+ \to \mathbb{C}^{m\times p}$ continuously differentiable, and complex matrices A_{11}, A_{12}, A_{21}, A_{22}, $K(\alpha)$, X_0, and Y_0 of appropriate dimensions. Notice that the matrix exponential (1) is a fundamental matrix of the ODE system given by (15) and (16), which is why the study of (15), (16) will be useful in the later development.

By Fact 1, the unique solutions to (15) and (16) (considered individually) are, for all $t \ge 0$,

$$X(t) = e^{(A_{11}-K(\alpha))t}X_0 + \int_0^t e^{(A_{11}-K(\alpha))(t-\tau)}A_{12}Y(\tau)\,d\tau \tag{17}$$

$$Y(t) = e^{A_{22}t}Y_0 + \int_0^t e^{A_{22}(t-\tau)}A_{21}X(\tau)\,d\tau. \tag{18}$$

Lemmas 2 and 3 below treat the solutions (17) and (18) for different initial conditions in the limit as the log norm of $-K(\alpha)$ tends to negative infinity. To establish these two lemmas, the following Gronwall-type inequality is used:

Lemma 1 (adapted from [10], Theorem 1.9): Let $v(t)$, $a(t)$, $b(t)$ be real-valued, nonnegative, continuous functions on $J = [t_0, t_1]$. Let $\kappa(t, s)$ be a real-valued, nonnegative, continuous function for $t_0 \le s \le t \le t_1$, and suppose

$$v(t) \le a(t) + b(t)\int_{t_0}^t \kappa(t, s)v(s)\,ds, \quad t \in J.$$

Then

$$v(t) \le \bar a(t)\exp\Big(\bar b(t)\int_{t_0}^t \bar\kappa(t, s)\,ds\Big), \quad t \in J,$$

where $\bar a(t) := \sup_{\tau\in[t_0,t]} a(\tau)$, $\bar b(t) := \sup_{\tau\in[t_0,t]} b(\tau)$, and $\bar\kappa(t, s) := \sup_{\tau\in[s,t]} \kappa(\tau, s)$.

Lemma 2: Consider the solutions (17) and (18) with the initial conditions $X_0 = I$ and $Y_0 = 0$. If $\lim_{\alpha\to\infty} \mu(-K(\alpha)) = -\infty$, then for $t > 0$,

$$\lim_{\alpha\to\infty} X(t) = 0 \quad \text{and} \quad \lim_{\alpha\to\infty} Y(t) = 0. \tag{19}$$

Proof: Since $\lim_{\alpha\to\infty} \mu(-K(\alpha)) = -\infty$, there exists an $\alpha_0 \in \mathbb{R}$ such that, for all $\alpha \ge \alpha_0$,

$$\mu(A_{11} - K(\alpha)) \le \|A_{11}\| + \mu(-K(\alpha)) < 0, \tag{20}$$

$$\mu(A_{11} - K(\alpha)) - \|A_{22}\| < -1. \tag{21}$$

In the following, we consider sufficiently large α such that $\alpha \ge \alpha_0$.

Substituting (17) into (18) and using the initial conditions $X_0 = I$ and $Y_0 = 0$ yields

$$\begin{aligned} Y(t) = &\int_0^t e^{A_{22}(t-\tau)}A_{21}e^{(A_{11}-K(\alpha))\tau}\,d\tau \\ &+ \int_0^t\!\!\int_0^\tau e^{A_{22}(t-\tau)}A_{21}e^{(A_{11}-K(\alpha))(\tau-s)}A_{12}Y(s)\,ds\,d\tau \\ = &\int_0^t e^{A_{22}(t-\tau)}A_{21}e^{(A_{11}-K(\alpha))\tau}\,d\tau \\ &+ \int_0^t\!\!\int_s^t e^{A_{22}(t-\tau)}A_{21}e^{(A_{11}-K(\alpha))(\tau-s)}A_{12}Y(s)\,d\tau\,ds, \end{aligned} \tag{22}$$

where the order of integration in the last term was interchanged. This is valid by Fubini's theorem, [11, Prop. 5.36], and the facts that the integrand is continuous, and the integration region can be expressed in either of the two ways: $\{(\tau, s) : 0 \le \tau \le t, 0 \le s \le \tau\}$ or $\{(\tau, s) : 0 \le s \le t, s \le \tau \le t\}$.

Using (9), (10), Fact 2, and submultiplicativity of the induced matrix norm, we obtain the inequality

$$\begin{aligned}
&\|Y(t)\| \\
&\le \|A_{21}\| \int_0^t \|e^{A_{22}(t-\tau)}\| \|e^{(A_{11}-K(\alpha))\tau}\| \, d\tau + \|A_{21}\| \|A_{12}\| \\
&\quad \times \int_0^t \int_s^t \|e^{A_{22}(t-\tau)}\| \|e^{(A_{11}-K(\alpha))(\tau-s)}\| \|Y(s)\| \, d\tau \, ds \\
&\le \|A_{21}\| \int_0^t e^{\|A_{22}\|(t-\tau)} e^{\mu(A_{11}-K(\alpha))\tau} \, d\tau + \|A_{21}\| \|A_{12}\| \\
&\quad \times \int_0^t \int_s^t e^{\|A_{22}\|(t-\tau)} e^{\mu(A_{11}-K(\alpha))(\tau-s)} \|Y(s)\| \, d\tau \, ds \qquad (23) \\
&= a(t) + \int_0^t \kappa(t,s) \|Y(s)\| \, ds, \qquad (24)
\end{aligned}$$

where

$$\begin{aligned}
a(t) &:= \|A_{21}\| \int_0^t e^{\|A_{22}\|(t-\tau)} e^{\mu(A_{11}-K(\alpha))\tau} \, d\tau, \\
\kappa(t,s) &:= \|A_{21}\| \|A_{12}\| \int_s^t e^{\|A_{22}\|(t-\tau)} e^{\mu(A_{11}-K(\alpha))(\tau-s)} \, d\tau.
\end{aligned}$$

Applying Lemma 1 to (24) yields, for all $t \ge 0$,

$$\|Y(t)\| \le \bar{a}(t) \exp\left(\int_0^t \bar{\kappa}(t,s) \, ds \right), \qquad (25)$$

where $\bar{a}(t) = \sup_{\tau \in [0,t]} a(\tau)$ and $\bar{\kappa}(t,s) = \sup_{\tau \in [s,t]} \kappa(\tau, s)$.

Next, we derive bounds for $a(t)$, $\bar{a}(t)$ and $\kappa(t,s)$, $\bar{\kappa}(t,s)$ using the properties (20), (21). First,

$$\begin{aligned}
a(t) &= \|A_{21}\| e^{\|A_{22}\|t} \int_0^t e^{(\mu(A_{11}-K(\alpha)) - \|A_{22}\|)\tau} \, d\tau \\
&= \frac{\|A_{21}\|}{\xi(\alpha)} \big(e^{\|A_{22}\|t} - \underbrace{e^{\mu(A_{11}-K(\alpha))t}}_{\in(0,1] \text{ by } (20)} \big) \\
&\le \frac{\|A_{21}\|}{\xi(\alpha)} e^{\|A_{22}\|t} = \frac{M_1(t)}{\xi(\alpha)},
\end{aligned}$$

where $\xi(\alpha) := \|A_{22}\| - \mu(A_{11} - K(\alpha)) > 1$ by (21), and $M_1(t) := \|A_{21}\| e^{\|A_{22}\|t} \ge 0$ is a continuous function in t. Therefore,

$$\begin{aligned}
\bar{a}(t) &= \sup_{\tau \in [0,t]} a(\tau) \le \sup_{\tau \in [0,t]} \frac{\|A_{21}\|}{\xi(\alpha)} e^{\|A_{22}\|\tau} \\
&= \frac{\|A_{21}\|}{\xi(\alpha)} e^{\|A_{22}\|t} = \frac{M_1(t)}{\xi(\alpha)}. \qquad (26)
\end{aligned}$$

Similarly, we obtain a bound for $\kappa(t,s)$. With $s \le t$,

$$\begin{aligned}
\kappa(t,s) &= \|A_{21}\| \|A_{12}\| e^{\|A_{22}\|t} e^{-\mu(A_{11}-K(\alpha))s} \\
&\quad \times \int_s^t e^{(\mu(A_{11}-K(\alpha)) - \|A_{22}\|)\tau} \, d\tau \\
&= \frac{\|A_{21}\| \|A_{12}\|}{\xi(\alpha)} \big(e^{\|A_{22}\|(t-s)} - \underbrace{e^{\mu(A_{11}-K(\alpha))(t-s)}}_{\in(0,1] \text{ by } (20)} \big) \\
&\le \frac{\|A_{21}\| \|A_{12}\|}{\xi(\alpha)} e^{\|A_{22}\|t} = \frac{M_2(t)}{\xi(\alpha)},
\end{aligned}$$

where $M_2(t) := \|A_{21}\| \|A_{12}\| e^{\|A_{22}\|t} \ge 0$ is a continuous function in t. Therefore,

$$\begin{aligned}
\bar{\kappa}(t,s) &= \sup_{\tau \in [s,t]} \kappa(\tau, s) \le \sup_{\tau \in [s,t]} \frac{\|A_{21}\| \|A_{12}\|}{\xi(\alpha)} e^{\|A_{22}\|\tau} \\
&= \frac{\|A_{21}\| \|A_{12}\|}{\xi(\alpha)} e^{\|A_{22}\|t} = \frac{M_2(t)}{\xi(\alpha)}. \qquad (27)
\end{aligned}$$

With (26) and (27), we can now bound (25),

$$\begin{aligned}
\|Y(t)\| &\le \frac{M_1(t)}{\xi(\alpha)} \exp\left(\int_0^t \frac{M_2(t)}{\xi(\alpha)} \, ds \right) \\
&= \frac{M_1(t)}{\xi(\alpha)} \exp\left(\frac{M_2(t)}{\xi(\alpha)} t \right) \\
&\le \frac{M_1(t)}{\xi(\alpha)} e^{tM_2(t)} = \frac{M(t)}{\xi(\alpha)}, \qquad (28)
\end{aligned}$$

where $M(t) := M_1(t) e^{tM_2(t)} \ge 0$ is continuous. Since $\lim_{\alpha \to \infty} \xi(\alpha) = \infty$, $\lim_{\alpha \to \infty} Y(t) = 0$ follows directly from (28). Furthermore, with (17) and $X_0 = I$,

$$\begin{aligned}
&\|X(t)\| \\
&\le e^{\mu(A_{11}-K(\alpha))t} + \|A_{12}\| \int_0^t \underbrace{e^{\mu(A_{11}-K(\alpha))(t-\tau)}}_{\in(0,1]} \|Y(\tau)\| \, d\tau \\
&\le e^{\mu(A_{11}-K(\alpha))t} + \frac{\|A_{12}\|}{\xi(\alpha)} \int_0^t M(\tau) \, d\tau. \qquad (29)
\end{aligned}$$

Therefore, $\lim_{\alpha \to \infty} X(t) = 0$ for $t > 0$. ■

Lemma 3: Consider the solutions (17) and (18) with the initial conditions $X_0 = 0$ and $Y_0 = I$. If $\lim_{\alpha \to \infty} \mu(-K(\alpha)) = -\infty$, then for $t > 0$,

$$\lim_{\alpha \to \infty} X(t) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} Y(t) = e^{A_{22}t}. \qquad (30)$$

Proof: The proof is essentially analogous to the proof of Lemma 2.

Let $\alpha \ge \alpha_0$ such that (20) and (21) hold. Substituting (18) into (17) and using the initial conditions $X_0 = 0$ and $Y_0 = I$ yields, after interchange of integration in the second term,

$$\begin{aligned}
X(t) &= \int_0^t e^{(A_{11}-K(\alpha))(t-\tau)} A_{12} e^{A_{22}\tau} \, d\tau \\
&\quad + \int_0^t \int_s^t e^{(A_{11}-K(\alpha))(t-\tau)} A_{12} e^{A_{22}(\tau-s)} A_{21} X(s) \, d\tau \, ds,
\end{aligned}$$

and, therefore,

$$\begin{aligned}
&\|X(t)\| \\
&\le \|A_{12}\| \int_0^t e^{\mu(A_{11}-K(\alpha))(t-\tau)} e^{\|A_{22}\|\tau} \, d\tau + \|A_{12}\| \|A_{21}\| \\
&\quad \times \int_0^t \int_s^t e^{\mu(A_{11}-K(\alpha))(t-\tau)} e^{\|A_{22}\|(\tau-s)} \|X(s)\| \, d\tau \, ds. \qquad (31)
\end{aligned}$$

Now, consider the substitutions $\tau \to t - \tau$ for the first term in (31) and $\tau \to t + s - \tau$ for the inner integral of the second term, which yields

$$\begin{aligned}
&\|X(t)\| \\
&\le \|A_{12}\| \int_0^t e^{\|A_{22}\|(t-\tau)} e^{\mu(A_{11}-K(\alpha))\tau} \, d\tau + \|A_{12}\| \|A_{21}\|
\end{aligned}$$

$$\times \int_0^t \int_s^t e^{\|A_{22}\|(t-\tau)} e^{\mu(A_{11}-K(\alpha))(\tau-s)} \|X(s)\|\, d\tau\, ds. \quad (32)$$

Comparing this inequality to (23), we find that (32) is obtained from (23) by the substitutions $\|Y(\cdot)\| \to \|X(\cdot)\|$, $\|A_{12}\| \to \|A_{21}\|$, and $\|A_{21}\| \to \|A_{12}\|$. Therefore, we can derive an upper bound on $\|X(t)\|$ the same way as in the proof of Lemma 2. Corresponding to (28) we get, for all $t \geq 0$,

$$\|X(t)\| \leq \frac{L(t)}{\xi(\alpha)},$$

where the continuous function $L(t) \geq 0$ is obtained from $M(t)$ by substituting $\|A_{12}\| \to \|A_{21}\|$ and $\|A_{21}\| \to \|A_{12}\|$. Thus, $\lim_{\alpha\to\infty} X(t) = 0$. Furthermore, with (18) and $Y_0 = I$,

$$\begin{aligned}\|Y(t) - e^{A_{22}t}\| &\leq \|A_{21}\| \int_0^t e^{\|A_{22}\|(t-\tau)} \|X(\tau)\|\, d\tau \\ &\leq \frac{\|A_{21}\|}{\xi(\alpha)} \int_0^t e^{\|A_{22}\|(t-\tau)} L(\tau)\, d\tau.\end{aligned}$$

Therefore, $\lim_{\alpha\to\infty} \|Y(t) - e^{A_{22}t}\| = 0$, and hence $\lim_{\alpha\to\infty} Y(t) = e^{A_{22}t}$. ■

Lemma 2 and 3 are used in the next section to establish one of the main results of this note (Theorem 3). The other two results (Theorem 4 and 5) presented in the next section are based on [4], [5], and, in particular on:

Theorem 1 ([4], Theorem 1): Let $A, B \in \mathbb{C}^{n\times n}$. Then $e^{(A+B/\varepsilon)t}$ converges pointwise as $\varepsilon \to 0^+$ for $t > 0$ if and only if B is semistable. If B is semistable, then

$$\lim_{\varepsilon\to 0^+} e^{(A+B/\varepsilon)t} = e^{(I-BB^{\mathrm{D}})At}(I - BB^{\mathrm{D}}). \quad (33)$$

Theorem 2 ([5], Theorem 1): Suppose $\operatorname{Index} C = 1$ and C is semistable. Then

$$e^{(A+B/\varepsilon + C/\varepsilon^r)t}, \quad r > 1, \quad (34)$$

converges as $\varepsilon \to 0^+$ for an $r \geq 2$, for all $t > 0$, if and only if $[B; C]$ is semistable. Suppose $[B; C]$ is semistable. If $r > 2$, then (34) converges to

$$e^{[[A;C];[B;C]]t}(I - [B; C]^{\mathrm{D}}[B; C])(I - C^{\mathrm{D}}C); \quad (35)$$

if $r = 2$, then the limit of (34) is the same as (35), except that a term

$$-[[BC^D B; C]; [B; C]]t \quad (36)$$

is added into the exponential.

IV. Main Results

This section establishes conditions on $K(\alpha)$ that guarantee (2). In Sec. IV-A, a sufficient condition is presented that is based on the log norm of $-K(\alpha)$ (Theorem 3). In Sec. IV-B and IV-C, we consider the case when $K(\alpha)$ has a particular polynomial structure; namely

$$K(\alpha) = K_0 + \alpha K_1 \quad \text{and} \quad (37)$$

$$K(\alpha) = K_0 + \alpha K_1 + \alpha^r K_2,\ r \geq 2, \quad (38)$$

respectively. For the affine case (37), a necessary and sufficient condition is derived (Theorem 4); and for (38), we present a sufficient condition (Theorem 5). Following each theorem, we give numerical examples in order to illustrate the applicability of the different results.

If $K(\alpha)$ represents a feedback controller gain such as in (5), equations (37) and (38) describe explicit parametrizations of the controller gain via the scalar tuning parameter α. If one seeks to analyze a controller parametrization that is not explicitly given as a function of α, the log norm condition can be useful, as shall be illustrated later in Example 1. The specific functional dependencies considered in (37) and (38) (affine and polynomial) correspond to those that are also studied in [4], [5] (therein as polynomials in $1/\varepsilon$, cf. Theorem 1 and 2).

A. Condition Based on the Log Norm of $K(\alpha)$

A sufficient condition for (2) is the log norm of $-K(\alpha)$ becoming arbitrarily small. This result is obtained by considering the matrix ODE that is solved uniquely by the matrix exponential (1) and then applying Lemmas 2 and 3 of Sec. III.

Theorem 3: Let $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{smallmatrix}\right] \in \mathbb{C}^{(n+m)\times(n+m)}$, and let $K : \mathbb{R} \to \mathbb{C}^{n\times n}$ be a matrix function of the real parameter α. If $\lim_{\alpha\to\infty} \mu(-K(\alpha)) = -\infty$, then (2) holds for all $t > 0$.

Proof: By Fact 1, the matrix exponential

$$\mathcal{X}(t) := \exp\left(\begin{bmatrix} A_{11} - K(\alpha) & A_{12} \\ A_{21} & A_{22}\end{bmatrix} t\right)$$

is the unique solution to the matrix ODE

$$\dot{\mathcal{X}}(t) = \begin{bmatrix} A_{11} - K(\alpha) & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \mathcal{X}(t),\ t \geq 0,\ \mathcal{X}(0) = I. \quad (39)$$

Note that $\mathcal{X} : \mathbb{R}^+ \to \mathbb{C}^{(n+m)\times(n+m)}$ is continuously differentiable. By subdividing $\mathcal{X}(t)$ into block matrices of appropriate dimensions,

$$\mathcal{X}(t) = \begin{bmatrix} \mathcal{X}_{11}(t) & \mathcal{X}_{12}(t) \\ \mathcal{X}_{21}(t) & \mathcal{X}_{22}(t)\end{bmatrix},$$

we can write (39) equivalently as

$$\begin{bmatrix} \dot{\mathcal{X}}_{11}(t) \\ \dot{\mathcal{X}}_{21}(t)\end{bmatrix} = \begin{bmatrix} A_{11} - K(\alpha) & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \begin{bmatrix} \mathcal{X}_{11}(t) \\ \mathcal{X}_{21}(t)\end{bmatrix},\ \begin{bmatrix} \mathcal{X}_{11}(0) \\ \mathcal{X}_{21}(0)\end{bmatrix} = \begin{bmatrix} I \\ 0\end{bmatrix}, \quad (40)$$

$$\begin{bmatrix} \dot{\mathcal{X}}_{12}(t) \\ \dot{\mathcal{X}}_{22}(t)\end{bmatrix} = \begin{bmatrix} A_{11} - K(\alpha) & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \begin{bmatrix} \mathcal{X}_{12}(t) \\ \mathcal{X}_{22}(t)\end{bmatrix},\ \begin{bmatrix} \mathcal{X}_{12}(0) \\ \mathcal{X}_{22}(0)\end{bmatrix} = \begin{bmatrix} 0 \\ I\end{bmatrix}. \quad (41)$$

Notice that (40) and (41) represent the matrix ODEs considered in Lemmas 2 and 3, respectively. Using these two lemmas, we therefore conclude that, for $t > 0$,

$$\begin{aligned}&\lim_{\alpha\to\infty} \exp\left(\begin{bmatrix} A_{11} - K(\alpha) & A_{12} \\ A_{21} & A_{22}\end{bmatrix} t\right) \\ &\quad = \lim_{\alpha\to\infty} \begin{bmatrix} \mathcal{X}_{11}(t) & \mathcal{X}_{12}(t) \\ \mathcal{X}_{21}(t) & \mathcal{X}_{22}(t)\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & e^{A_{22}t}\end{bmatrix}.\end{aligned}$$

■

We illustrate Theorem 3 with the following examples.

Example 1 (Discretization and high-gain feedback): Consider the feedback control example from the introduction given by (3)–(5), and assume a diagonal structure for the feedback gain $K(\alpha)$ with diagonal elements $k_i(\alpha)$. Suppose we are interested in a discrete-time description of the closed-loop system (3)–(5) at a rate $T > 0$. The discretized system reads

$$\begin{bmatrix} x_{\mathrm{f}}(t+T) \\ x_{\mathrm{s}}(t+T) \end{bmatrix} = \exp\left(\begin{bmatrix} A_{11} - K(\alpha) & A_{12} \\ A_{21} & A_{22} \end{bmatrix} T \right) \begin{bmatrix} x_{\mathrm{f}}(t) \\ x_{\mathrm{s}}(t) \end{bmatrix}. \quad (42)$$

Now, assume $k_i(\alpha) \geq \alpha$, that is, the individual controller gains are at least as large as α. Then,

$$\lim_{\alpha\to\infty} \mu(-K(\alpha)) = \lim_{\alpha\to\infty} \max_i \big(-k_i(\alpha) \big) \leq \lim_{\alpha\to\infty} -\alpha = -\infty, \quad (43)$$

and, by Theorem 3, (42) becomes, in the limit as $\alpha \to \infty$,

$$x_{\mathrm{f}}(t+T) = 0 \quad (44)$$

$$x_{\mathrm{s}}(t+T) = e^{A_{22}T} x_{\mathrm{s}}(t)\,; \quad (45)$$

that is, the slow and fast dynamics are decoupled.

In [1] and [3, p. 65], Theorem 3 is applied to control systems that are more general than (3)–(5), where, in particular, the controller (5) is modified to track a reference input changing at the rate T. The obtained discrete-time model of the control system is then used to design an outer-loop controller that commands the reference input to the modified controller (5), which then acts as the inner-loop controller of the cascaded control system.

Remark 1: Note that the function $K(\alpha)$ is not given explicitly in Example 1. The estimate $k_i(\alpha) \geq \alpha$ with the diagonal structure of $K(\alpha)$ is enough to verify the condition of Theorem 3. In contrast, the convergence results in Sec. IV-B and IV-C require an explicit description of $K(\alpha)$.

Example 2: Consider (1) with

$$A = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right], \; K(\alpha) = \begin{bmatrix} \alpha & \alpha^2 \\ 0 & \alpha \end{bmatrix}, \text{ and } \; t = 1.$$

Notice that $-K(\alpha)$ is stable for all $\alpha > 0$ (both eigenvalues are $-\alpha$), and that both eigenvalues go to negative infinity as $\alpha \to \infty$. But $\lim_{\alpha\to\infty} \mu(-K(\alpha)) = \lim_{\alpha\to\infty} \max\{-\alpha + \frac{1}{2}\alpha^2, -\alpha - \frac{1}{2}\alpha^2\} = \infty$, and the limit of (1) for $\alpha \to \infty$ is (can be computed using [7, Fact 11.14.2])

$$\left[\begin{array}{cc|c} 0 & 0 & -1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right],$$

which is clearly different from (2) in the (1,2)-block.

Remark 2: The preceding example shows that it does not suffice for (2) to hold that the eigenvalues of $-K(\alpha)$ tend to negative infinity.

B. $K(\alpha)$ *Affine*

We study the limit of (1) with affine $K(\alpha)$ as in (37). The following result provides a necessary and sufficient condition for (2). It is obtained using Theorem 1.

Theorem 4: Let $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right] \in \mathbb{C}^{(m+n)\times(m+n)}$, and let $K(\alpha) = K_0 + \alpha K_1$ with $K_0, K_1 \in \mathbb{C}^{n\times n}$ and $\alpha \in \mathbb{R}$. Then, (2) holds for $t > 0$ if and only if $-K_1$ is stable.

Proof: We first prove sufficiency. Let

$$\tilde{A} := \begin{bmatrix} A_{11} - K_0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \tilde{B} := \begin{bmatrix} -K_1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (46)$$

Since $-K_1$ is stable, $\tilde{B}$ is semistable, and it follows from Theorem 1 (by substituting $1/\varepsilon$ with α) that

$$\lim_{\alpha\to\infty} e^{(\tilde{A}+\alpha\tilde{B})t} = \lim_{\varepsilon\to 0^+} e^{(\tilde{A}+\tilde{B}/\varepsilon)t} = e^{(I-\tilde{B}\tilde{B}^{\mathrm{D}})\tilde{A}t}(I - \tilde{B}\tilde{B}^{\mathrm{D}}). \quad (47)$$

Since $-K_1$ is stable, it is invertible and $\tilde{B}^{\mathrm{D}} = \left[\begin{smallmatrix} -K_1^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Hence, we have

$$\begin{aligned} e^{(I-\tilde{B}\tilde{B}^{\mathrm{D}})\tilde{A}t} &= \exp\left(\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} - K_0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} t \right) \\ &= \exp\left(\begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} t \right) = \begin{bmatrix} I & 0 \\ * & e^{A_{22}t} \end{bmatrix}, \quad (48) \end{aligned}$$

where the last equality follows from [7, Fact 11.14.2], and $*$ is a placeholder left unspecified. Therefore, we get from (47)

$$\begin{aligned} \lim_{\alpha\to\infty} e^{(\tilde{A}+\alpha\tilde{B})t} &= e^{(I-\tilde{B}\tilde{B}^{\mathrm{D}})\tilde{A}t}(I - \tilde{B}\tilde{B}^{\mathrm{D}}) \quad (49) \\ &= \begin{bmatrix} I & 0 \\ * & e^{A_{22}t} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & e^{A_{22}t} \end{bmatrix}, \quad (50) \end{aligned}$$

which completes the sufficiency part of the proof.

For the necessity proof, assume (2) holds. First notice that, for the limit $\lim_{\alpha\to\infty} e^{(\tilde{A}+\alpha\tilde{B})t} = \lim_{\varepsilon\to 0^+} e^{(\tilde{A}+\tilde{B}/\varepsilon)t}$ to exist, it follows from Theorem 1 that $\tilde{B}$ is semistable. From the definition of $\tilde{B}$ in (46), it can be seen that this implies that $-K_1$ is semistable, which further implies that

$$\mathrm{spec}(-K_1) \subset \mathrm{OLHP} \cup \{0\}. \quad (51)$$

From $\tilde{B}$ being semistable and Theorem 1, it follows that (47) holds. Hence, the limit in (47) is equal to the limit in (2):

$$e^{(I-\tilde{B}\tilde{B}^{\mathrm{D}})\tilde{A}t}(I - \tilde{B}\tilde{B}^{\mathrm{D}}) = \begin{bmatrix} 0 & 0 \\ 0 & e^{A_{22}t} \end{bmatrix}. \quad (52)$$

Now, let

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} := e^{(I-\tilde{B}\tilde{B}^{\mathrm{D}})\tilde{A}t}. \quad (53)$$

Using $\tilde{B}^{\mathrm{D}} = \left[\begin{smallmatrix} -K_1^{\mathrm{D}} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and (53), it follows from (52) (by considering the first block column) that

$$\begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} (I - K_1 K_1^{\mathrm{D}}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (54)$$

Since the matrix exponential is nonsingular [7, Prop. 11.2.8], E in (53) is nonsingular, and $\left[\begin{smallmatrix} E_{11} \\ E_{21} \end{smallmatrix}\right]$ has full column rank. Therefore, (54) implies $(I - K_1K_1^{\mathrm{D}}) = 0 \Leftrightarrow K_1K_1^{\mathrm{D}} = I$. From this and the rank formula [7, Lemma 2.5.2] $n = \mathrm{rank}(I) = \mathrm{rank}(K_1K_1^{\mathrm{D}}) \leq \min\{\mathrm{rank}(K_1), \mathrm{rank}(K_1^{\mathrm{D}})\} \leq n$, it follows that K_1 has full rank. Thus, also $-K_1$ has full rank, which implies $0 \notin \mathrm{spec}(-K_1)$, [7, Cor. 2.6.6, Prop. 5.5.20]. This and (51) imply $\mathrm{spec}(-K_1) \subset \mathrm{OLHP}$, i.e. $-K_1$ is stable. ■

Example 3: Consider

$$K(\alpha) = \alpha K_1 \quad \text{with} \quad K_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Then $-K_1$ is stable, and, by Theorem 4, (2) holds for any A.

Remark 3: For $K(\alpha)$ as in Example 3, we compute $\lim_{\alpha\to\infty} \mu(-K(\alpha)) = \lim_{\alpha\to\infty} \max\{0, -2\alpha\} = 0$. Example 3 hence shows that the condition in Theorem 3 is not a necessary condition.

C. $K(\alpha)$ *Polynomial*

We study the limit of (1) with $K(\alpha)$ as in (38); that is, compared to (37), $K(\alpha)$ possesses an additional power α^r with $r \geq 2$. A sufficient condition for convergence is derived using Theorem 2. The condition is different from the sufficient condition in Theorem 3 (one does not imply the other) as shall be pointed out later.

Theorem 5: Let $A = \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right] \in \mathbb{C}^{(m+n)\times(m+n)}$, and let $K(\alpha) = K_0 + \alpha K_1 + \alpha^r K_2$ with $K_0, K_1, K_2 \in \mathbb{C}^{n\times n}$ and $\alpha, r \in \mathbb{R}$, $r \geq 2$. If $-K_2$ is stable, then (2) holds for $t > 0$.

Proof: Let $\tilde{A}$, $\tilde{B}$ be as in (46), and let $\tilde{C} := \left[\begin{smallmatrix} -K_2 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Since $-K_2$ is stable, $\tilde{C}$ is semistable. Furthermore, $\operatorname{Index} \tilde{C} = 1$ since $\operatorname{rank}(K_2^2) = \operatorname{rank}(-K_2)$ ($-K_2$ has full rank). Thus, the assumptions of Theorem 2 are satisfied.

With $\tilde{C}^{\mathrm{D}} = \left[\begin{smallmatrix} -K_2^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$, we get $(I - \tilde{C}^{\mathrm{D}}\tilde{C}) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ and $[\tilde{B};\tilde{C}] = (I - \tilde{C}^{\mathrm{D}}\tilde{C})\tilde{B}(I - \tilde{C}^{\mathrm{D}}\tilde{C}) = 0$, which is semistable. Therefore, by Theorem 2, $\lim_{\alpha\to\infty} e^{(\tilde{A}+\alpha\tilde{B}+\alpha^r\tilde{C})t} = \lim_{\varepsilon\to 0^+} e^{(\tilde{A}+\tilde{B}/\varepsilon+\tilde{C}/\varepsilon^r)t}$ converges to the limit specified by (35) and (36) where A, B, C are replaced by $\tilde{A}$, $\tilde{B}$, $\tilde{C}$. We next compute the expressions (35) and (36).

From $[\tilde{B};\tilde{C}] = 0$, we get $(I - [\tilde{B};\tilde{C}]^{\mathrm{D}}[\tilde{B};\tilde{C}]) = I - 0^{\mathrm{D}}0 = I$. Furthermore,

$$[\tilde{A};\tilde{C}] = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} - K_0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix},$$

and, hence,

$$[[\tilde{A};\tilde{C}];[\tilde{B};\tilde{C}]] = [\tilde{A};\tilde{C}] = \begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}. \tag{55}$$

Using these results, expression (35) yields the desired limit in (2),

$$e^{[[\tilde{A};\tilde{C}];[\tilde{B};\tilde{C}]]t}(I - [\tilde{B};\tilde{C}]^{\mathrm{D}}[\tilde{B};\tilde{C}])(I - \tilde{C}^{\mathrm{D}}\tilde{C}) = \begin{bmatrix} 0 & 0 \\ 0 & e^{A_{22}t} \end{bmatrix}.$$

Since

$$[\tilde{B}\tilde{C}^{\mathrm{D}}\tilde{B};\tilde{C}] = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -K_1 K_2^{-1} K_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = 0,$$

expression (36) is 0. ■

Example 4: Consider

$$K(\alpha) = -\alpha I + \alpha^2 K_2 \quad \text{with} \quad K_2 = \begin{bmatrix} 0.1 & 1 \\ 0 & 0.1 \end{bmatrix}.$$

Then $-K_2$ is stable, and, by Theorem 5, (2) holds for any A. Notice that the instability of $-K_1 = I$ is irrelevant. From $\lim_{\alpha\to\infty} \mu(-K(\alpha)) = \lim_{\alpha\to\infty} \max\{\alpha + \frac{2\alpha^2}{5}, \alpha - \frac{3\alpha^2}{5}\} = \infty$ we see that Theorem 3 is not helpful here.

Example 5: Consider

$$K(\alpha) = \alpha K_1 + \alpha^r K_2 \quad \text{with} \quad 1 < r < 2$$
$$\text{and} \quad K_1 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix},\; K_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Theorem 5 is not helpful here, since $r < 2$ (neither is Theorem 2). But $\mu(-K(\alpha)) = \max\{-\alpha, 2\alpha - \alpha^r\} \to -\infty$ as $\alpha \to \infty$; hence, (2) follows from Theorem 3.

Remark 4: Examples 4 and 5 show that there are problems with $K(\alpha) = K_0 + \alpha K_1 + \alpha^r K_2$ which are covered by Theorem 3, but not by Theorem 5; and vice versa. In general, both theorems provide sufficient conditions for different problem classes.

V. Concluding Remarks

The three theorems presented in this technical note guarantee the limiting property (2) of the matrix exponential (1); essentially, a "large enough" $K(\alpha)$ in the (1,1)-block forces all but the (2,2)-block of the matrix exponential to tend to zero in the limit. Theorem 3 states a sufficient condition for (2) based on the log norm of $-K(\alpha)$, and Theorems 4 and 5 provide sufficient conditions for $K(\alpha)$ having a particular polynomial form. For the affine case (Theorem 4), the obtained condition is also necessary.

Theorems 4 and 5 herein are obtained using the results by Campbell et al. [4], [5]. Theorem 3, however, is obtained independently of those results. Its method of proof is based on the matrix differential equation that is solved uniquely by the matrix exponential (1), and on bounding its solution using a Gronwall-type inequality. In contrast, Campbell et al. make use of Cauchy's integral formula to prove their result in [4], for example.

The numerical examples herein were chosen to highlight specific mathematical properties of the results. The limiting property of the matrix exponential has been applied to practical examples in [1] (an inverted pendulum) and [3] (a balancing cube). Therein, Theorem 3 is used to derive a time-scale separation algorithm that computes a discrete-time model representing the plant dynamics under high-gain feedback on some of the plant's states. This model is then used to design a stabilizing outer-loop controller.

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